

Regularization of spherically symmetric evolution codes in numerical relativity

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The lack of regularity of geometric variables at the origin is often a source of serious problem for spherically symmetric evolution codes in numerical relativity. One usually deals with this by restricting the gauge and solving the hamiltonian constraint for the metric. Here we present a generic algorithm for dealing with the regularization of the origin that can be used directly on the evolution equations and that allows very general gauge choices. Our approach is similar in spirit to the one introduced by Arbona and Bona for the particular case of the Bona-Masso formulation. However, our algorithm is more general and can be used with a wide variety of evolution systems.

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I. INTRODUCTION

When developing spherically symmetric codes in numerical relativity, the coordinate singularity at the origin can be a source of serious problems caused by the lack of regularity of the geometric variables there. The problem arises because of the presence of terms in the evolution equations that go as $1/r$ near the origin. Regularity of the metric (essentially local flatness) guarantees the exact cancellations of such terms at the origin, thus ensuring well-behaved solutions. This exact cancellation, however, though certainly true for analytical solutions, usually fails to hold for numerical solutions. One then finds that the $1/r$ terms do not cancel and the numerical solution becomes ill-behaved near $r = 0$: it not only fails to converge there, but can easily turn out to be violently unstable in just a few timesteps.

The usual way to deal with this problem is to use the so-called *areal* (or *radial*) gauge, where the radial coordinate r is chosen in such a way that the proper area of spheres of constant r is always $4\pi r^2$. If, moreover, one also chooses a vanishing shift vector one ends up in the standard *polar/areal* gauge [1, 2], for which the lapse is forced to satisfy a certain ordinary differential equation in r . The name *polar* comes from the fact that for this gauge choice there is only one non-zero component of the extrinsic curvature tensor, namely K_{rr} [1]. In the polar/areal gauge the problem of achieving the exact cancellation of the $1/r$ terms is reduced to imposing the boundary condition $g_{rr} = 1$ at $r = 0$, which can be easily done if one solves for g_{rr} from the hamiltonian constraint (which in this case is an ordinary differential equation in r) and ignores its evolution equation. If one does this in vacuum, one ends up inevitably with Minkowski spacetime in the usual coordinates (one can also recover Schwarzschild by working in isotropic coordinates and factoring out the conformal factor analytically). Of course, in the presence of matter, one can still have truly non-trivial dynamics.

The main drawback of the standard approach is that the gauge choice has been completely exhausted. In particular, the polar/areal gauge can not penetrate apparent horizons, since inside an apparent horizon it is impossible to keep the areas of spheres fixed without a non-trivial shift vector. The polar/areal gauge has, nevertheless, been used successfully even in the study of critical collapse to a black hole, where the presence of the black hole is identified by the familiar “collapse of the lapse” even if no apparent horizon can be found [3].

Still, one would like to have a way of dealing with the regularity issue that allows more generic gauge choices to be made, either because one is interested in studying the region inside an apparent horizon, or because one wants to test interesting gauge conditions in the simple case of spherical symmetry. Because of this we have developed a general regularization technique that can be used directly on the Einstein evolution equations.

Our regularization method is similar in spirit, if not in detail, to the one presented by Arbona and Bona in [4]. The main difference being that the approach of Arbona and Bona was tied to the use of the Bona-Masso evolution system [5, 6, 7, 8, 9], while our algorithm is much more general.

A final point deserves notice. Spherically symmetric evolution codes that involve eternal black holes usually ignore the regularity problem. For example, one can excise the black hole interior and eliminate $r = 0$ from the numerical grid. But even if one does not use excision, for a black hole $r = 0$ is not a regular point but rather a compactification of the asymptotic infinity on the other side of the Einstein-Rosen bridge. This compactification introduces geometric factors that compensate the $1/r$ terms making the equations regular even at $r = 0$. This means that, contrary to what one would naively expect, in spherical symmetry it is easier to evolve eternal black holes (at least for some time) than it is to evolve regular spacetimes.

This paper is organized as follows. In Sec. II we discuss the regularity conditions that the metric functions must satisfy at the origin of spherical coordinates, and we show which terms need to be regularized in the Einstein equations. Section III describes our regularization

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algorithm in a generic way. In Sec. IV we present example of how to regularize some specific formulations of the Einstein equations, and show some numerical examples. We conclude in section V.

II. REGULARITY CONDITIONS

We start by writing the general form of the spatial metric in spherical symmetry as

$$dl^2 = A(r, t)dr^2 + r^2 B(r, t)d\Omega^2, \quad (2.1)$$

with A and B positive metric functions and $d\Omega^2$ the solid angle element: $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Notice that we have already factored out the r^2 dependency of the angular metric functions. This has the advantage of making explicit the dependency on r of geometric quantities and makes the regularization procedure easier.

As we will deal with the Einstein equations in first order form, we will introduce the auxiliary quantities:

$$D_A := \partial_r \ln A, \quad D_B := \partial_r \ln B. \quad (2.2)$$

We will also work with the mixed components of the extrinsic curvature: $K_A := K_r^r, K_B := K_\theta^\theta = K_\phi^\phi$.

There are in fact two different types of regularity conditions that the variables $\{A, B, D_A, D_B, K_A, K_B\}$ must satisfy at $r = 0$. The first type of conditions are simple those imposed by the requirement that the different variables should be well defined at the origin, and imply the following behavior for small r :

$$A \sim A^0 + \mathcal{O}(r^2), \quad (2.3)$$

$$B \sim B^0 + \mathcal{O}(r^2), \quad (2.4)$$

$$D_A \sim \mathcal{O}(r), \quad (2.5)$$

$$D_B \sim \mathcal{O}(r), \quad (2.6)$$

$$K_A \sim K_A^0 + \mathcal{O}(r^2), \quad (2.7)$$

$$K_B \sim K_B^0 + \mathcal{O}(r^2), \quad (2.8)$$

with $\{A^0, B^0, K_A^0, K_B^0\}$ perhaps functions of time, but not of r . These regularity conditions are in fact quite easy to implement numerically. For example, one can use a finite differencing grid that staggers the origin, and then obtain data on the fictitious point at $r = -\Delta r/2$ by demanding for $\{A, B, K_A, K_B\}$ to be even functions at $r = 0$ and for $\{D_A, D_B\}$ to be odd.

It is the second type of regularity conditions that is more troublesome. In order to see the problem, we will first write the Arnowitt-Deser-Misner (ADM) equations [10, 11] in first order form for the case of spherical symmetry. The evolution equations are (in the case of zero shift)

$$\partial_t A = -2\alpha A K_A, \quad (2.9)$$

$$\partial_t B = -2\alpha B K_B, \quad (2.10)$$

$$\partial_t D_A = -2\alpha [K_A D_\alpha + \partial_r K_A], \quad (2.11)$$

$$\partial_t D_B = -2\alpha [K_B D_\alpha + \partial_r K_B], \quad (2.12)$$

$$\begin{aligned} \partial_t K_A = & -\frac{\alpha}{A} \left[\partial_r (D_\alpha + D_B) + D_\alpha^2 - \frac{D_\alpha D_A}{2} \right. \\ & + \frac{D_B^2}{2} - \frac{D_A D_B}{2} - A K_A (K_A + 2K_B) \\ & \left. - \frac{1}{r} (D_A - 2D_B) \right], \end{aligned} \quad (2.13)$$

$$\begin{aligned} \partial_t K_B = & -\frac{\alpha}{2A} \left[\partial_r D_B + D_\alpha D_B + D_B^2 - \frac{D_A D_B}{2} \right. \\ & \left. - \frac{1}{r} (D_A - 2D_\alpha - 4D_B) - \frac{2(A-B)}{r^2 B} \right] \\ & + \alpha K_B (K_A + 2K_B) \end{aligned} \quad (2.14)$$

where α is the lapse function and $D_\alpha := \partial_r \ln \alpha$. The hamiltonian and momentum constraints take the form

$$\begin{aligned} \partial_r D_B = & \frac{1}{r^2 B} (A - B) + A K_B (2K_A + K_B) \\ & + \frac{1}{r} (D_A - 3D_B) + \frac{D_A D_B}{2} - \frac{3D_B^2}{4}, \end{aligned} \quad (2.15)$$

$$\partial_r K_B = (K_A - K_B) \left[\frac{1}{r} + \frac{D_B}{2} \right], \quad (2.16)$$

Since $\{D_\alpha, D_A, D_B\}$ go as r near the origin, terms of the type $D_{\{\alpha, A, B\}}/r$ are regular and represent no problem. However, we see that both in the hamiltonian constraint and in the evolution equation for K_B there is a term of the form $(A-B)/r^2$, while in the momentum constraint there is a term of the form $(K_A - K_B)/r$. Given the behavior of these variables near $r = 0$ these terms would seem to blow up at the origin. The reason why this does not in fact happen is that, near the origin, we must also ask for the extra regularity conditions

$$A - B \sim \mathcal{O}(r^2), \quad K_A - K_B \sim \mathcal{O}(r^2), \quad (2.17)$$

that is

$$A^0 = B^0, \quad K_A^0 = K_B^0. \quad (2.18)$$

It is not difficult to understand where these conditions come from. They are just a consequence of the fact that space must remain locally flat at $r = 0$. This local flatness condition implies that, near $r = 0$, it must be possible to write the metric as

$$dl^2_{R \sim 0} = dR^2 + R^2 d\Omega^2, \quad (2.19)$$

with R a radial coordinate that measures proper distance from the origin. A local transformation of coordinates from R to r then takes the metric into the form

$$dl^2_{r \sim 0} = \left(\frac{dR}{dr} \right)^2_{r=0} (dr^2 + r^2 d\Omega^2), \quad (2.20)$$

which implies that $A^0 = B^0$ and, since this must hold for all time, also that $K_A^0 = K_B^0$.

It turns out that it is not trivial to implement numerically both the symmetry regularity conditions (2.3)-(2.8) and the local flatness regularity conditions (2.17) at the same time. The reason for this is that at $r = 0$ we now have three boundary conditions for just two variables: both the derivatives of A and B must vanish, plus A and B must be equal to each other (and the same thing must happen for K_A and K_B). The boundary conditions for the exact equations are, of course, also over-determined, but in that case the consistency of the equations implies that if they are satisfied initially they remain satisfied for all time. In the numerical case, however, this is not true due to truncation errors, and very rapidly (typically within one or two time steps) one of the three boundary conditions fails to hold. It is easy to convince oneself that simply ignoring one condition and imposing the other two does not work. If we impose the zero derivative condition and ignore the $A = B$ condition, then the $(A - B)/r^2$ term in the evolution equations rapidly becomes singular. On the other hand, if we impose the $A = B$ condition and ignore one of the symmetry conditions, then we introduce an inconsistency with the finite difference version of the evolution equations, since for finite Δr it is very difficult to guarantee that the difference between $\partial_t K_A$ and $\partial_t K_B$ approaches zero at the origin. This inconsistency then very rapidly causes large (and non-convergent) gradients to develop near the origin. In the following section we will introduce an algorithm that successfully regularizes the numerical evolution equations near $r = 0$.

As a final comment, from the above equations we can also very easily see why the polar/areal gauge has no serious regularity problem. In that gauge we have $B = 1$ by construction. If we now impose the boundary condition $A(r = 0) = 1$, and solve for $A(r)$ by integrating the hamiltonian constraint with $B = 1$ and $D_B = 0$ (ignoring the evolution equations), then the $(A - B)/r^2$ term causes no trouble.

III. REGULARIZATION ALGORITHM

In Ref. [4], Arbona and Bona developed a regularization technique for the spherically symmetric version of the Bona-Masso (BM) evolution system [5, 6, 7, 8, 9]. Their technique is based on redefining the auxiliary dynamical variable V_r that is part of the standard BM formulation in a way that allows them to absorb the $(A - B)/r^2$ terms and reduces the regularization problem to applying the correct boundary condition to V_r at $r = 0$.

Since we are interested in developing a generic regularization technique, we will start from the ADM equations from the previous section. However, we will take the idea from the regularization technique of Arbona and Bona of introducing an auxiliary variable that will allow us to absorb the problematic terms. Adding an auxiliary variable seems to us to be the more straightforward way of solving

the problem of having over-determined boundary conditions: the extra boundary condition will be imposed on the auxiliary variable. We will then define the variable,

$$\lambda := \frac{1}{r} \left(1 - \frac{A}{B} \right). \quad (3.1)$$

Notice that, if the local flatness regularity conditions (2.17) are satisfied, then the variable λ has the following behavior at the origin

$$\lambda \sim \mathcal{O}(r), \quad (3.2)$$

which, as mentioned above, can easily be imposed numerically using a grid that staggers the origin, and asking for λ to be odd across $r = 0$.

The hamiltonian constraint now becomes

$$\begin{aligned} \partial_r D_B &= \frac{\lambda}{r} + AK_B (2K_A + K_B) \\ &+ \frac{1}{r} (D_A - 3D_B) + \frac{D_A D_B}{2} - \frac{3D_B^2}{4}, \end{aligned} \quad (3.3)$$

and the evolution equation for K_B becomes

$$\begin{aligned} \partial_t K_B &= -\frac{\alpha}{2A} \left[\partial_r D_B + D_\alpha D_B + D_B^2 - \frac{D_A D_B}{2} \right. \\ &\quad \left. - \frac{1}{r} (D_A - 2D_\alpha - 4D_B - 2\lambda) \right] \\ &+ \alpha K_B (K_A + 2K_B). \end{aligned} \quad (3.4)$$

As mentioned, the problem terms have now been transformed into λ/r , which will be well behaved as long as λ is odd at $r = 0$. The momentum constraint still has a term $(K_A - K_B)/r$, but this should cause no trouble since it does not feed back into the evolution equations (one can always multiply the momentum constraint with r before evaluating it). Of course, multiples of the momentum constraint are typically added to the evolution equations in order to build hyperbolic formulations, and we will discuss how to deal with that term in such a case below.

There is one other ingredient that needs to be added: an evolution equation for λ . This can be obtained directly from its definition:

$$\partial_t \lambda = \frac{2\alpha A}{B} \left(\frac{K_A - K_B}{r} \right). \quad (3.5)$$

The last evolution equation clearly has the dangerous $(K_A - K_B)/r$ term, but this term can be removed with the help of the momentum constraint (2.16) to find

$$\partial_t \lambda = \frac{2\alpha A}{B} \left[\partial_r K_B - \frac{D_B}{2} (K_A - K_B) \right], \quad (3.6)$$

which is now regular at the origin.

The regularized first order ADM evolution equations are then (2.9)-(2.13), with (2.14) replaced by (3.4), plus the evolution equation for λ given by (3.6).

Having regularized the standard ADM equations, the question arises of how to regularize alternative formulations where multiples of the constraints can be added to the evolution equations in a number of ways. Adding multiples of the hamiltonian constraint represents no problem, as the introduction of the variable λ already regularized this constraint, as seen in (3.3). The momentum constraint, however, is not regularized as it still includes the term $(K_A - K_B)/r$. One could try to play the same game as before and introduce yet another variable to absorb this term. However, we will now show that this is not really necessary.

Let us then consider some arbitrary first order formulation of the Einstein evolution equations in spherical symmetry that has the generic form

$$\partial_t u_i = q_i(u, v), \quad (3.7)$$

$$\partial_t v_i = M_i^j(u) \partial_r v_j + p_i(u, v), \quad (3.8)$$

where $u = (A, B, \lambda)$ and $v = (D_A, D_B, K_A, K_B)$. The source terms q and p are assumed not to depend on derivatives of any of the fields. The formulation might be hyperbolic or not, depending on the characteristic structure of the matrix M . We will assume that one has arrived at such a formulation by adding multiples of the hamiltonian and momentum constraints to the evolution equations for the v 's. This means that one can expect that the source terms p_i will in general contain terms proportional to $(K_A - K_B)/r$. We will then rewrite the evolution equations for the v_i as

$$\partial_t v_i = M_i^j(u) \partial_r v_j + p'_i(u, v) + \frac{f_i(u)}{r} (K_A - K_B). \quad (3.9)$$

Here we are assuming that the coefficient $f_i(u)$ of the $(K_A - K_B)/r$ terms depends of the u 's, but not on the v 's, which will typically be the case. Using now equation (3.5) we find

$$\partial_t v_i = M_i^j(u) \partial_r v_j + p'_i(u, v) + \frac{f_i(u)B}{2\alpha A} \partial_t \lambda, \quad (3.10)$$

which implies

$$\begin{aligned} \partial_t \left(v_i - \frac{f_i(u)B}{2\alpha A} \lambda \right) &= M_i^j(u) \partial_r v_j + p'_i(u, v) \\ &\quad - \lambda \partial_t \left(\frac{f_i(u)B}{2\alpha A} \right). \end{aligned} \quad (3.11)$$

If we now define

$$v'_i := v_i - \frac{f_i(u)B}{2\alpha A} \lambda, \quad (3.12)$$

we can transform the last equation into

$$\partial_t v'_i = M_i^j(u) \partial_r v_j + p'_i(u, v) - \lambda F_i^t(u, v), \quad (3.13)$$

with $F_i^t(u, v) = \partial_t (f_i(u)B/2\alpha A)$. Notice that $F_i^t(u, v)$ so defined will involve no spatial derivatives of u 's or v 's.

The final step is to substitute the spatial derivative of v_j for that of v'_j to find

$$\begin{aligned} \partial_t v'_i &= M_i^j(u) \partial_r v'_j + p'_i(u, v) - \lambda F_i^t(u, v) \\ &\quad + \partial_r \left(\frac{f_i(u)B}{2\alpha A} \lambda \right) \\ &= M_i^j(u) \partial_r v'_j + p'_i(u, v) + \lambda (F_i^r(u, v) - F_i^t(u, v)) \\ &\quad + \left(\frac{f_i(u)B}{2\alpha A} \right) \partial_r \lambda, \end{aligned} \quad (3.14)$$

with $F_i^r(u, v) = \partial_r (f_i(u)B/2\alpha A)$. Using now the fact that

$$\partial_r \lambda = -\frac{1}{r} \left[\lambda + \frac{A}{B} (D_A - D_B) \right], \quad (3.15)$$

we finally find

$$\begin{aligned} \partial_t v'_i &= M_i^j(u) \partial_r v'_j + p'_i(u, v) + \lambda (F_i^r(u, v) - F_i^t(u, v)) \\ &\quad - \frac{f_i(u)B}{2\alpha A r} \left[\lambda + \frac{A}{B} (D_A - D_B) \right]. \end{aligned} \quad (3.16)$$

This last equation is now regular, and has precisely the same characteristic structure as the original system. What we have done is transform the original evolution equations for the v_i variables into evolution equations for the new v'_i variables for which the principal part terms are the same and the source terms are regular. Notice that typically only some of the $f_i(u)$ will be different from zero, so one does not need to transform all variables.

In the following section we will consider examples of how to regularize some specific systems of evolution equations.

IV. EXAMPLES

The regularized first order ADM equations were derived in the last section. Here we will examples of numerical evolutions using two different regularized systems.

In the numerical simulations we will take as initial data Minkowski spacetime in the usual coordinates, so that:

$$A = B = 1, \quad (4.1)$$

$$D_A = D_B = 0, \quad (4.2)$$

$$K_A = K_B = 0. \quad (4.3)$$

In order to have a non-trivial evolution, we will chose an initial lapse profile of the form:

$$\alpha(t=0) = 1 + r^2 \hat{C} e^{-\frac{(r-\hat{r})^2}{\hat{\sigma}^2}}, \quad (4.4)$$

that is, we add a small gaussian contribution to the initial Minkowski lapse. We will then evolve the lapse using a Bona-Masso (BM) slicing condition [8], so the evolution equation for the lapse will be:

$$\partial_t \alpha = -\alpha^2 f(\alpha) (K_A + 2K_B). \quad (4.5)$$

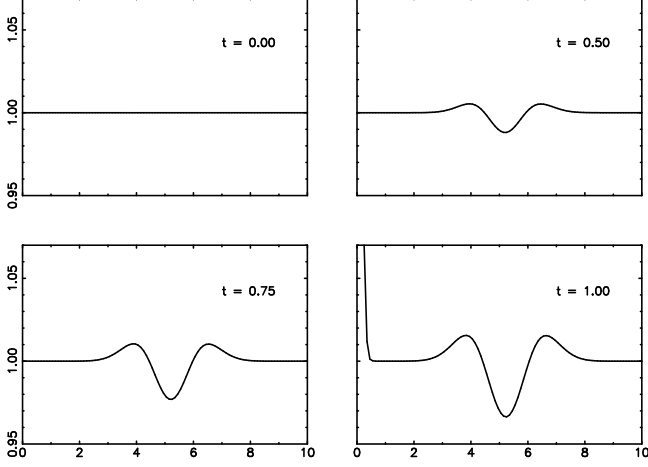


FIG. 1: System I not regularized. The plots show the evolution of the metric function A at different times. Note the spike at $r = 0$ for $t = 1$.

In the simulations shown below we have taken the gaussian parameters to be: $\hat{C} = 0.001, \hat{r} = 5.0, \hat{\sigma} = 1.0$. We have also restricted ourselves to harmonic slicing, that is, $f(\alpha) = 1$.

A. System I

As a first example we will build a hyperbolic system starting from the ADM equations and the BM slicing condition. We construct this system by using the Hamiltonian and Momentum constraints to remove the terms proportional to $\partial_r D_B$ and $\partial_r K_B$ from the evolution equations of K_A and D_α respectively. The resulting system can be easily shown to be strongly hyperbolic.

Figures 1 and 2 show the evolution of the radial metric A and lapse function α using the system described above, with no regularization. Note that both plots show a spike at $r = 0$ for times $t \approx 1$.

Next we look at the regularized case. As described above, we first introduce the auxilliary variable

$$\lambda = \frac{1}{r} \left(1 - \frac{A}{B} \right). \quad (4.6)$$

Also, since we have used the momentum constraint to modify the evolution equation for D_α , we will need to replace this variable with

$$U_\alpha := D_\alpha + \frac{B\lambda}{A}. \quad (4.7)$$

The set of variables to be evolved is then:

$$\{\alpha, A, B, U_\alpha, D_A, D_B, K_A, K_B, \lambda\}. \quad (4.8)$$

The evolution equations for these variables take the form:

$$\partial_t \alpha = -\alpha^2 f(K_A + 2K_B), \quad (4.9)$$

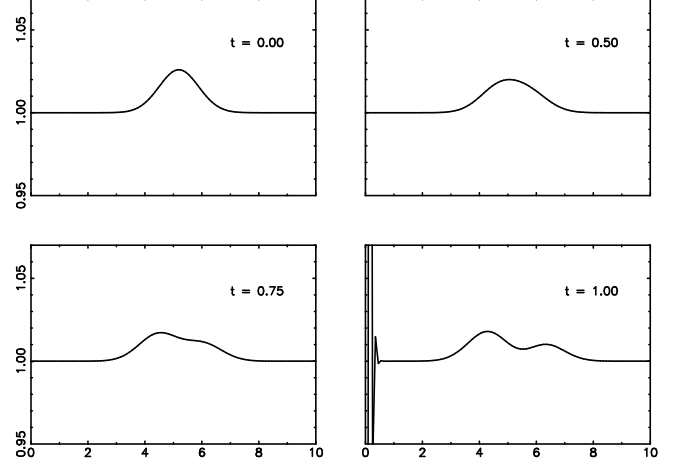


FIG. 2: System I not regularized. The plots show the evolution of the lapse function α at different times. Note the spike at $r = 0$ for $t = 1$.

$$\partial_t A = -2\alpha A K_A, \quad (4.10)$$

$$\partial_t B = -2\alpha B K_B, \quad (4.11)$$

$$\begin{aligned} \partial_t U_\alpha = & -\alpha f \partial_r K_A + \alpha (K_A + 2K_B) \left[\frac{f^2 B \lambda}{A} \right. \\ & \left. - U_\alpha (f + \alpha f') \right] + \alpha f (K_B - K_A) \\ & \left(D_B - \frac{2B\lambda}{A} \right), \end{aligned} \quad (4.12)$$

$$\partial_t D_A = -2\alpha \left[\partial_r K_A + K_A \left(U_\alpha - \frac{fB\lambda}{A} \right) \right], \quad (4.13)$$

$$\partial_t D_B = -2\alpha \left[\partial_r K_B + K_B \left(U_\alpha - \frac{fB\lambda}{A} \right) \right], \quad (4.14)$$

$$\begin{aligned} \partial_t K_A = & -\frac{\alpha}{A} \partial_r U_\alpha - \frac{\alpha}{A} \left\{ \left(U_\alpha - \frac{fB\lambda}{A} \right) \left[U_\alpha - \frac{D_A}{2} \right. \right. \\ & - \frac{B\lambda}{A} (f + \alpha f') \left. \right] + \frac{fB\lambda}{A} (D_A - D_B) - \frac{D_B^2}{4} \\ & + A (K_B^2 - K_A^2) - \frac{1}{r} \left[\lambda + D_B \right. \\ & \left. \left. - f \left(\frac{B\lambda}{A} + D_A - D_B \right) \right] \right\}, \end{aligned} \quad (4.15)$$

$$\begin{aligned} \partial_t K_B = & -\frac{\alpha}{2A} \partial_r D_B + \frac{\alpha}{2A} \left\{ - \left(U_\alpha - \frac{fB\lambda}{A} \right) D_B - D_B^2 \right. \\ & + \frac{D_A D_B}{2} + 2AK_B (K_A + 2K_B) + \frac{1}{r} \left[D_A - 2U_\alpha \right. \\ & \left. \left. + \frac{2fB\lambda}{A} - 4D_B - 2\lambda \right] \right\}, \end{aligned} \quad (4.16)$$

$$\partial_t \lambda = \frac{2\alpha A}{B} \partial_r K_B + \frac{\alpha A D_B}{B} (K_B - K_A). \quad (4.17)$$

with $f' := df/d\alpha$.

Figures 3 and 4 shown again the evolution of A and α

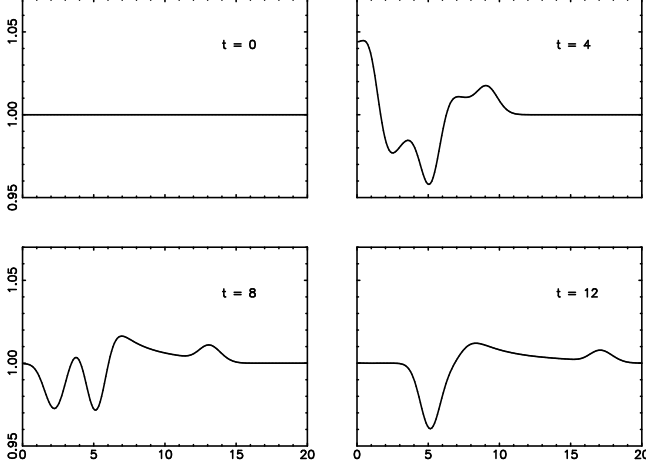


FIG. 3: System I regularized. The plots show the evolution of the metric function A at different times.

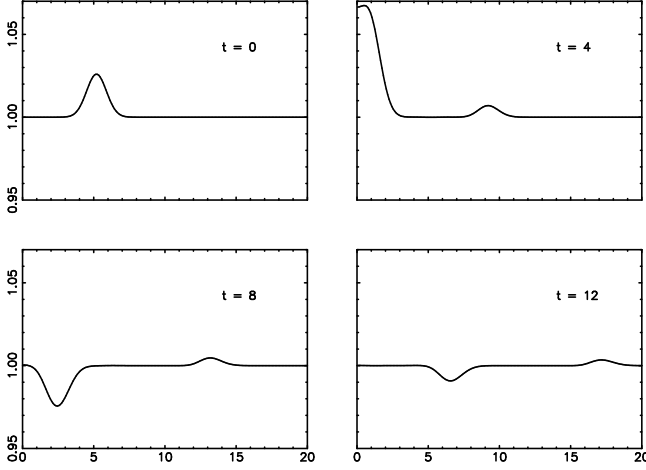


FIG. 4: System I regularized. The plots show the evolution of the lapse function α at different times.

for the regularized case. The system can now be evolved for very long times and the origin remains well behaved. The plots show the evolution up to $t = 12$.

B. System II

Our second example is again a hyperbolic system built from the ADM equations and the BM slicing condition, but now instead of using D_A and K_A as fundamental variables, we have used $\tilde{D} = D_A - 2D_B$ and $K \equiv \text{tr}K := K_A + 2K_B$. In order to make the system hyperbolic we remove the terms proportional to $\partial_r D_B$ and $\partial_r K_B$ from the evolution equations of K and \tilde{D} using the constraints. We then obtain a new strongly hyperbolic system different from System I above.

For the regularization procedure we introduce again

the variable λ . Since in this case we have used the momentum constraint to modify the evolution equation for \tilde{D} , we need to replace this variable with

$$\tilde{U} := \tilde{D} - \frac{4B\lambda}{A}. \quad (4.18)$$

The final set of dynamical variables is then

$$\{\alpha, A, B, D_\alpha, \tilde{U}, D_B, K, K_B, \lambda\}, \quad (4.19)$$

and their evolution equations are:

$$\partial_t \alpha = -\alpha^2 f K, \quad (4.20)$$

$$\partial_t A = 2\alpha A(2K_B - K), \quad (4.21)$$

$$\partial_t B = -2\alpha B K_B, \quad (4.22)$$

$$\partial_t D_\alpha = -\partial_r(\alpha f K), \quad (4.23)$$

$$\begin{aligned} \partial_t \tilde{U} = & -2\partial_r(\alpha K) + 4\alpha D_B(K - 3K_B) \\ & + 8\alpha \left[D_\alpha K_B + \frac{B\lambda}{A}(3K_B - K) \right], \end{aligned} \quad (4.24)$$

$$\partial_t D_B = -2\partial_r(\alpha K_B), \quad (4.25)$$

$$\begin{aligned} \partial_t K = & \alpha \left[-4K K_B + 6K_B^2 - \frac{2D_\alpha}{Ar} + K^2 \right. \\ & \left. + \frac{D_\alpha}{2A} \left(\tilde{U} + \frac{4\lambda B}{A} \right) - \frac{D_\alpha^2}{A} - \frac{\partial_r D_\alpha}{A} \right], \end{aligned} \quad (4.26)$$

$$\begin{aligned} \partial_t K_B = & \frac{\alpha}{Ar} \left[\frac{\tilde{U}}{2} + \frac{2\lambda B}{A} - D_B - \lambda - D_\alpha \right] \\ & + \frac{\alpha}{A} \left[-\frac{D_\alpha D_B}{2} - \frac{\partial_r D_B}{2} \right. \\ & \left. + \frac{D_B}{4} \left(\tilde{U} + \frac{4\lambda B}{A} \right) + A K K_B \right], \end{aligned} \quad (4.27)$$

$$\partial_t \lambda = \frac{2\alpha A}{B} \left[\partial_r K_B - \frac{D_B}{2}(K - 3K_B) \right]. \quad (4.28)$$

In this case there is no need to show any plots, as the numerical evolution behaves exactly in the way it did for System I, with the origin remaining well behaved.

V. DISCUSSION

Lack of regularity of geometric variables at the origin is often a problem for spherically symmetric evolution codes in numerical relativity. We have shown that the problem can be traced to the existence of two types of regularity conditions at the origin. In the first place, there are regularity conditions that guarantee that the variables are well defined at the origin. These conditions can be written down as a series of symmetry conditions at the origin for the different variables, and can be easily enforced in numerical simulations. However, there also exist regularity conditions related to the condition that spacetime must be locally flat at the origin. Together, all these regularity conditions imply that we have more conditions to satisfy at $r = 0$ than dynamical variables,

which means that numerically some regularity conditions will inevitably be violated. We have presented a generic regularization algorithm that is based on the introduction of an auxiliary variable that absorbs the problematic terms and on which we can impose the extra boundary conditions at $r = 0$. Our algorithm is similar in spirit,

if not in detail, to an algorithm presented by Arbona and Bona for the particular case of the Bona-Masso formulation [4]. We have also shown the effectiveness of our algorithm on a couple of different formulations of the evolution equations.

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